I don't even know what a knot is!

Topologically Trivial Legendrian Knots

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Main Theorem

Let L and L' be two topologically trivial Legendrian knots in a tight contact 3-manifold.
If $tb(L) = tb(L')$ and $r(L) = r(L')$ then L and L' are Legendrian isotopic.

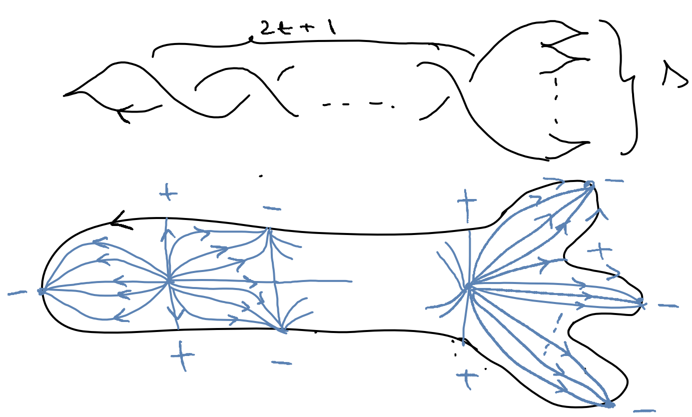
Proof Strategy

Let *L* be a Legendrian knot bounding an embedded disk *D*.

1. Perturb the foliation
2. Build a tree
3. Define a front projection and a foliation
4. Modify the tree

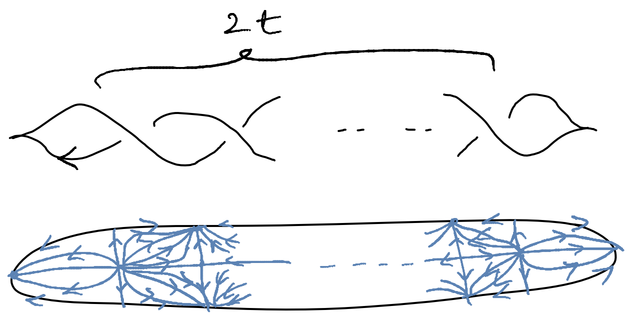
Catalog of Wavefronts

- r=-s < 0, tb = -(2t+1+s)



Catalog of Wavefronts

- r=s > 0, tb = -(2t+1+s)
  - Reverse orientations in the previous slide
- r=0, tb = -(2t+1)



Step1: Perturb the foliation

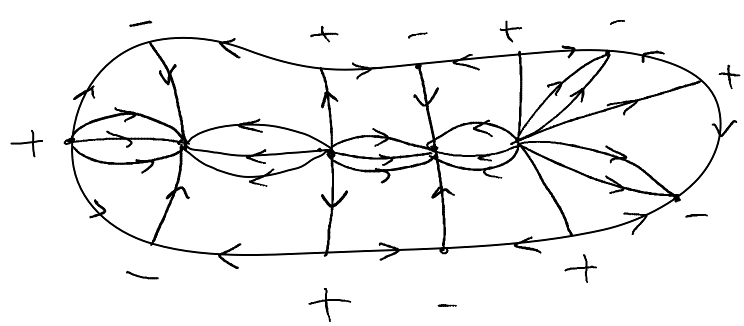
Goal: Given a spanning disk *D* of *L*, perform a C^0-small perturbation of *D* to obtain a spanning disk *D'* of *L* with foliation in elliptic form.

1. Just h+ and e- on boundary
2. Just h+ and e- on boundary and just e+ and h- on interior
3. Mostly h+ and h- on boundary, just e+ and e- on interior

Elliptic Foliation

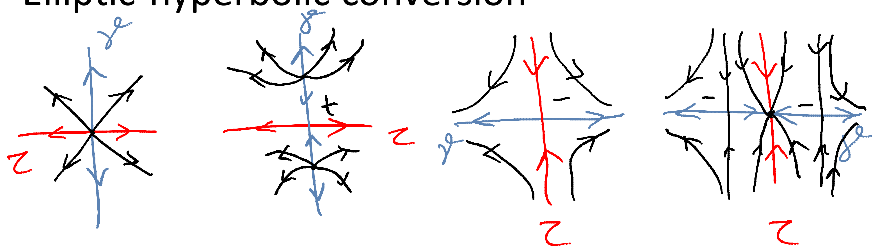
- Signs of boundary singularities alternate
- Boundary singularities connect only with their direct neighbors on the boundary and interior singularities
- All interior singularities are elliptic
- Interior singularities connect to at least two boundary hyperbolic singularities

Elliptic Foliation



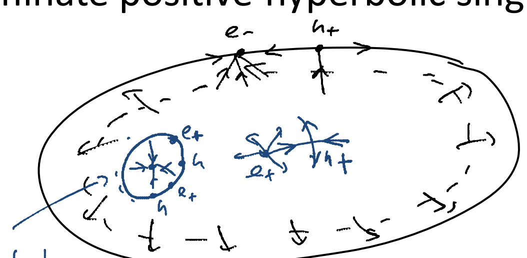
Just h+ and e- on boundary

- If *tb(L)=t* then there is a C^0-small perturbation of *D* such that there are exactly *2t* singularities on the boundary and they have alternating signs.
- Elliptic-hyperbolic conversion



Just h- and e+ on interior

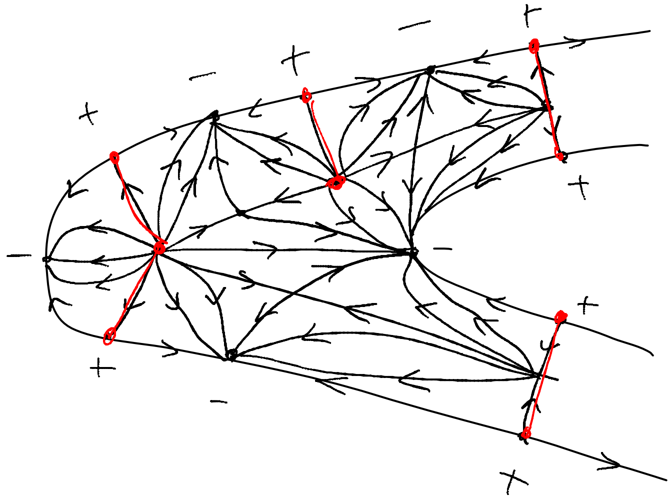
- Destroy hyperbolic-hyperbolic connections
- Eliminate negative elliptic singularities
- Eliminate positive hyperbolic singularities



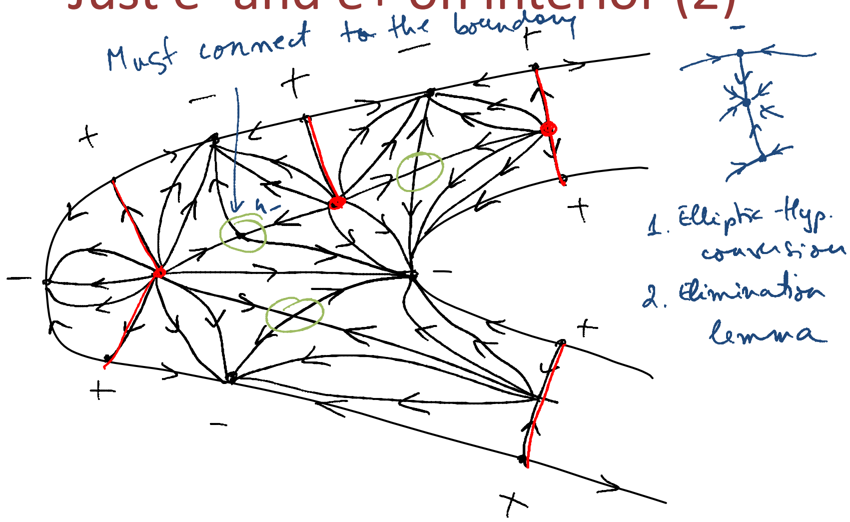
Use the elimination lemma

Must have an h-
otherwise we have a cycle

Just e- and e+ on interior (1)

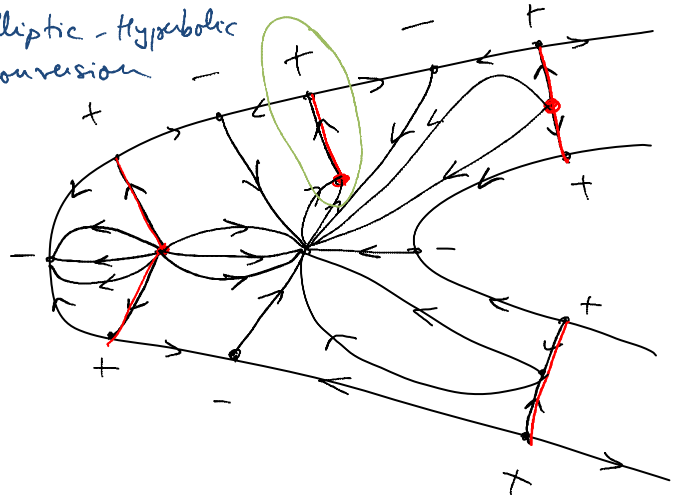


Just e- and e+ on interior (2)

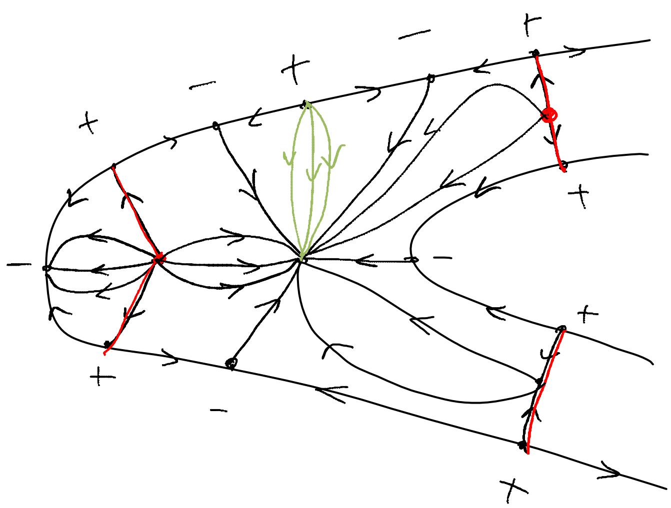


Just e- and e+ on interior (3)

Elliptic - Hyperbolic conversion



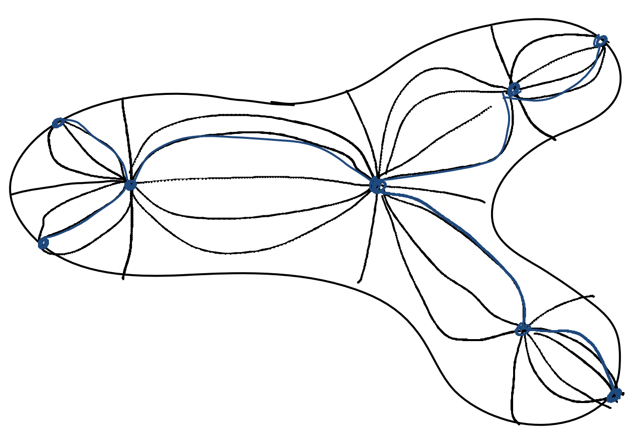
Just e- and e+ on interior (4)



Step 2: Build a Tree

- Skeleton of the foliation
  - Vertices - interior elliptic points
  - Edges – representative arcs
- Extended skeleton of the foliation
  - New vertices – elliptic boundary points
  - New edges – representative arcs
- Signed trees 
- Have an acceptable planar embedding

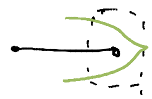
Extended Skeleton



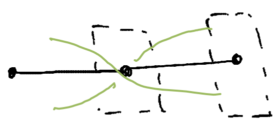
Build an wavefront

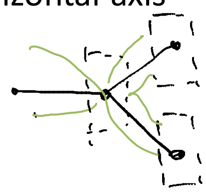
- Choose disjoint neighborhoods of vertices
- Leftmost vertex

- End vertex



- Otherwise – replace the subtree to the right by a reflection of it in the horizontal axis





Recap

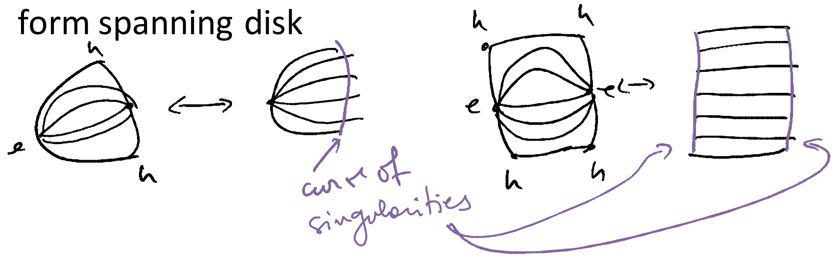
- Start with Legendrian knot L spanned by the embedded disk D
- Perturb D to have an elliptic foliation
- Get an embedded Legendrian tree T (extended skeleton)
- Given a planar embedding of T build a front projection W_T

Claim: The lift of W_T bounds an embedded disk whose foliation is elliptic and diffeomorphic to the elliptic foliation of D.

Forget about L (1)

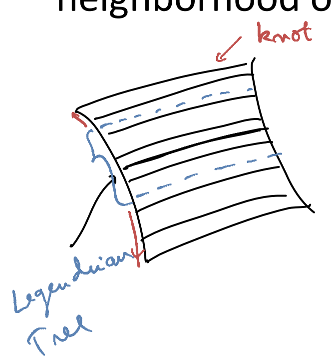
Suppose Legendrian knots L and L' bound D and D' with diffeomorphic characteristic foliations in elliptic form.
Then L and L' are Legendrian isotopic.

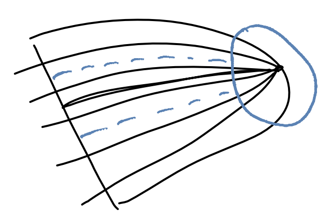
Convert the elliptic form spanning disk to exceptional form spanning disk



Forget about L (2)

Isotopy supported in the complement of small neighborhood of end vertices.



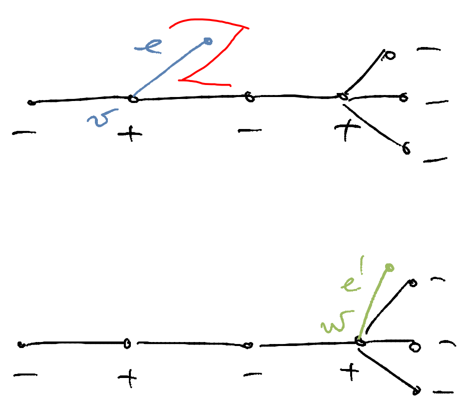


Forget about L (3)

Use Elliptic Pivot Lemma to extend the isotopy to the entire disk.

We can assume that in a neighborhood of the elliptic point we can choose cylindrical coordinates (ρ, ϕ, z) and the contact form is $dz + \rho^2 d\phi$. Let L_c be a piecewise-smooth Legendrian curve in the horizontal plane consisting of two rays $\phi = 0$ and $\phi = c$.
For any $\epsilon > 0$ there exists a Legendrian isotopy $\hat{L}_c$, $c \in (0, \pi]$ such that $\hat{L}_\pi = L_\pi$ and for all $c \in (0, \pi]$ the curve $\hat{L}_c$ coincides with L_c outside of the ϵ-neighborhood of the origin.

Step 4: Modify the Tree



Step 4: Modify the Tree