# Lie Algebras: Abstract Theory of Weights

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These notes are based on chapter 13 in 'Introduction to Lie Algebras and Representation Theory', by James E. Humphreys.

Let $\Phi$ be a root system in an euclidean space $E$, with Weyl group $\mathcal{W}$.

Recall, a subset $\Phi$ of the euclidean space $E$ is called a **root system** in $E$ if the following axioms are satisfied:
(R1) $\Phi$ is finite, spans $E$ and it does not contain 0.
(R2) If $\alpha \in \Phi$, the only multiples of $\alpha$ in $\Phi$ are $\pm\alpha$.
(R3) If $\alpha \in \Phi$, the reflexion $\sigma_\alpha$ leaves $\Phi$ invariant.
(R4) If $\alpha, \beta \in \Phi$ then $< \beta, \alpha > \in \mathbb{Z}$.

For each $\alpha \in E$ we define $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Let $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$.

**Lemma 1:** $\Phi^\vee$ is a root system.

*Proof.* I will check the four axioms of the root system.
(R1) Since $|\Phi^\vee| = |\Phi|$, then $\Phi^\vee$ is finite. Since $\Phi$ spans $E$ and $\alpha^\vee$ is a non-zero scalar multiple of $\alpha$ for all $\alpha \in \Phi$, then $\Phi^\vee$ spans $E$ and $0 \notin \Phi^\vee$.
(R2) Since $(-\alpha)^\vee = \frac{2(-\alpha)}{((-\alpha),(-\alpha))} = -\frac{2\alpha}{(\alpha,\alpha)} = -\alpha^\vee$, then the only multiples of $\alpha^\vee$ are $\pm\alpha^\vee$.
(R3) I need to show that if $\alpha^\vee \in \Phi^\vee$ then the reflection $\sigma_{\alpha^\vee}$ leaves $\Phi^\vee$ invariant. Let $\alpha$ in $\Phi$, then for all $\beta \in \Phi$, $\sigma_\alpha(\beta) \in \Phi$. Let $\beta \in \Phi$.

$$\begin{aligned}
\sigma_{\alpha^\vee}(\beta^\vee) &= \beta^\vee - \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)}\alpha^\vee \\
&= \frac{2\beta}{(\beta,\beta)} - \frac{2(\frac{2\beta}{(\beta,\beta)}, \frac{2\alpha}{(\alpha,\alpha)})}{(\frac{2\alpha}{(\alpha,\alpha)}, \frac{2\alpha}{(\alpha,\alpha)})}\frac{2\alpha}{(\alpha,\alpha)} \\
&= \frac{2\beta}{(\beta,\beta)} - \frac{4(\beta,\alpha)\alpha}{(\beta,\beta)(\alpha,\alpha)} \\
&= \frac{2}{(\beta,\beta)}\left(\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha)\right) \\
&= \frac{2}{(\beta,\beta)}\sigma_\alpha(\beta)
\end{aligned}$$

But,

$$
\begin{aligned}
(\sigma_\alpha(\beta), \sigma_\alpha(\beta)) &= \left(\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \right) \\
&= (\beta,\beta) - \frac{2(\beta,\alpha)(\beta,\alpha)}{(\alpha,\alpha)} - \frac{2(\beta,\alpha)(\alpha,\beta)}{(\alpha,\alpha)} + \frac{4(\beta,\alpha)^2(\alpha,\alpha)}{(\alpha,\alpha)^2} \\
&= (\beta,\beta) - \frac{4(\beta,\alpha)^2}{(\alpha,\alpha)} + \frac{4(\beta,\alpha)^2}{(\alpha,\alpha)} \\
&= (\beta,\beta)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sigma_{\alpha^\vee}(\beta^\vee) &= \frac{2}{(\beta,\beta)}\sigma_\alpha(\beta) \\
&= \frac{2}{(\sigma_\alpha(\beta), \sigma_\alpha(\beta))}\sigma_\alpha(\beta) \\
&= (\sigma_\alpha(\beta))^\vee
\end{aligned}
$$

Thus, $\sigma_{\alpha^\vee}(\beta^\vee) \in \Phi^\vee$.
(R4) I need to show that $< \alpha^\vee, \beta^\vee > \in \mathbb{Z}$. We have:

$$
\begin{aligned}
< \alpha^\vee, \beta^\vee > &= \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)} \\
&= \frac{2(\frac{2\beta}{(\beta,\beta)}, \frac{2\alpha}{(\alpha,\alpha)})}{(\frac{2\alpha}{(\alpha,\alpha)}, \frac{2\alpha}{(\alpha,\alpha)})} \\
&= \frac{2(\beta,\alpha)}{(\beta,\beta)} \\
&= \frac{2(\alpha,\beta)}{(\beta,\beta)} \\
&= < \beta, \alpha >
\end{aligned}
$$

Since $\Phi$ is a root system, then $< \beta, \alpha > \in \mathbb{Z}$. Therefore $< \alpha^\vee, \beta^\vee > \in \mathbb{Z}$. □

**Lemma 2:** If $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ is a basis of $E$ contained in $\Phi$ such that for all $\alpha \in \Phi$, $\alpha = \sum_{i=1}^{l} k_i\alpha_i$ all $k_i$ are nonnegative or all $k_i$ are nonpositive, then $\Delta$ is a base of $E$.

*Proof. Step 1:* Find $\gamma$ regular such that $(\gamma, \alpha_i) > 0$ for all $1 \leq i \leq l$.
For each $1 \leq i \leq l$, let $P_i$ be the hyperplane generated by $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_l$. Let $\delta_i$ be the projection of $\alpha_i$ on the orthogonal complement of $P_i$. I will show that $(\delta_i, \alpha_i) > 0$. Let $v_1, \ldots, v_{l-1}$ be an orthonormal basis of $P_i$. Then $v_1, \ldots, v_{l-1}, w$ where $w = \frac{\delta_i}{|\delta_i|}$ is an orthonormal basis of $E$. Let $v = \alpha_i$. By Pythagoras theorem, $(v, v) =$

$\sum_{i=0}^{l-1}(v,v_i)^2 + (v,w)^2$, therefore $(v,v) - \sum_{i=0}^{l-1}(v,v_i)^2 \geq 0$. But, $(\delta_i, \alpha_i) = |\delta_i|(w,v) = |\delta_i|(v - \sum_{j=1}^{l-1}(v,v_j)v_j, v) = |\delta_i|\left((v,v) - \sum_{j=1}^{l-1}(v,v_j)^2\right) \geq 0$. Since $\alpha_i \notin P_i$, then $(\delta_i,\alpha_i) > 0$.
Let $\gamma = \sum_{i=1}^{l} r_i\delta_i$ where $r_i > 0$. Since $(\delta_i, \alpha_j) = 0$ for all $i \neq j$, then $(\gamma, \alpha_i) = r_i(\delta_i, \alpha_i) > 0$.
Let $\alpha \in \Phi$. Then $\alpha = \sum_{i=1}^{l} k_i\alpha_i$ all $k_i$ are nonnegative or all $k_i$ are nonpositive. Thus $(\gamma, \alpha) = \sum_{i=1}^{l} k_i(\gamma, \alpha_i)$. Since not all $k_i$ are zero and $(\gamma, \alpha_i) > 0$ and all $k_i$ are nonnegative or all $k_i$ are nonpositive, then $(\gamma, \alpha) \neq 0$. Therefore $\gamma \notin P_\alpha$. Since $\alpha$ was arbitrary, then $\gamma \notin \cup_{\alpha\in\Phi} P_\alpha$. By the definition, $\gamma$ is regular.
*Step 2:* $\Delta$ is a base of $\Phi$.
We know from a theorem in 10.1, that the set $\Delta(\gamma)$ of indecomposable elements of $\Phi^+(\gamma) = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}$ is a base of $\Phi$. Thus, it is enough to show that the elements of $\Delta$ are indecomposable. Assume $\alpha_i \in \Delta$ is decomposable, that is $\alpha_i = \beta_1 + \beta_2$, where $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Since $\Delta$ is a basis of $E$, then $\beta_1 = \sum_{j=1}^{l} k_j\alpha_j$ and $\beta_1 = \sum_{j=1}^{l} l_j\alpha_j$. Since $\Delta$ is a basis, then $k_j, l_j \in \mathbb{Z}$ for all $j$ and $1 = k_i + l_i$ and $k_j + l_j = 0$ for all $j \neq i$. We have two cases:
Case 1: $k_j = l_j = 0$ for all $i \neq j$
Then $\beta_1 = k_i\alpha_i$ and $\beta_2 = l_i\alpha_i$. Since $\Phi$ is a root system, $k_i, l_i \in \{\pm 1\}$. Then $k_i + l_i \in \{-2, 0, 2\}$. This contracts $k_i + l_i = 1$.
Case 2: $k_j \neq 0$ for some $j \neq i$
Since either all $k_i$ nonnegative or all $k_i$ nonpositive, and similarly for $l_i$, then then one set must be nonnegative and one nonpositive. Without loss of generality, assume $k_i \geq 0$ and $l_i \leq 0$. Then $(\gamma, \beta_2) = \sum_{i=1}^{l} l_i(\gamma, \alpha_i)$. Since $\gamma$ is regular, then $(\gamma, \alpha_i) > 0$ and $(\gamma, \beta_2) > 0$, we reached a contradiction.
Thus $\alpha_i$ is indecomposable. Therefore $\Delta = \Delta(\gamma)$. □

**Corollary:** If $\Delta$ is a base of $\Phi$, then $\Delta^\vee$ is a base of $\Phi^\vee$.

*Proof.* Since $(,)$ is positive definite, then $(\alpha, \alpha) > 0$ for all $\alpha \in E$. Since $\Delta$ is a base of $\Phi$, then $\alpha = \sum_{i=1}^{l} k_i\alpha_i$ all $k_i$ are nonnegative or all $k_i$ are nonpositive. Then

$\alpha^\vee = \sum_{i=1}^{l} k'_i \alpha_i^\vee$ where $k'_i = \frac{(\alpha_i,\alpha_i)}{(\alpha,\alpha)} k_i$ are all nonnegative or all nonpositive. By lemma 2, $\Delta^\vee$ is a base of $\Phi^\vee$.
A **weight** $\lambda$ is an element of the euclidean space $E$ such that $< \lambda, \alpha > \in \mathbb{Z}$ for all $\alpha \in \Phi$. Let $\Lambda$ denote the set of all weights. Since $<,>$ is linear in the first factor, then $\Lambda$ is a subgroup of $E$. By the axiom (R4) of the definition of the root system, $\Phi \subset \Lambda$. □

**Lemma 3:** Let $\Phi$ be a root system in an euclidean space $E$, with base $\Delta$. Let $\Lambda$ be the set of weights. Then $\lambda \in \Lambda$ iff $< \lambda, \alpha > \in \mathbb{Z}$ for all $\alpha \in \Delta$.

*Proof.* The forward implication follows from the definition of weight. Conversely, let $\alpha \in \Phi$. Since $< \lambda, \alpha > = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = (\lambda, \frac{2\alpha}{(\alpha,\alpha)}) = (\lambda, \alpha^\vee)$, then we need to show that $(\lambda, \alpha^\vee) \in \mathbb{Z}$. By the previous corollary, $\Delta^\vee$ is a base of $\Phi^\vee$, therefore $\alpha = \sum_{i=1}^{l} k_i \alpha_i^\vee$ where $k_i \in \mathbb{Z}$ for all $1 \leq i \leq l$. Thus $(\lambda, \alpha^\vee) = \sum_{i=1}^{l} k_i (\lambda, \alpha_i^\vee)$. By assumption $< \lambda, \alpha_i > \in \mathbb{Z}$ for all $1 \leq i \leq l$, thus $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$. Thus $(\lambda, \alpha^\vee) \in \mathbb{Z}$ as a sum of integers. □

The **root lattice** $\Lambda_r$ is the subgroup of $\Lambda$ generated by $\Phi$.

**Lemma 4:** $\Lambda_r$ is a lattice.

*Proof.* Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a base of $\Phi$. Then $\Delta$ is a basis of $E$. I will show that $\Lambda_r$ is the $\mathbb{Z}$-span of $\Delta$.
Let $\sum_{i=1}^{l} k_i \alpha_i$ be in the $\mathbb{Z}$-span of $\Delta$. By the axiom $(R4)$ of the root system, $< \alpha_i, \alpha_i > \in Z$. Thus $< \sum_{i=1}^{l} k_i \alpha_i, \alpha_j > = \sum_{i=1}^{l} k_i < \alpha_i, \alpha_j > \in \mathbb{Z}$ as the sum of integers. Therefore, by lemma 3, $\sum_{i=1}^{l} k_i \alpha_i$ is a weight, and it is in the subgroup generated by $\Delta \subset \Phi$.
Let $\lambda \in \Lambda_r$. Then $\lambda = \sum_{\alpha \in \Phi} k_\alpha \alpha$ where $k_\alpha \in \mathbb{Z}$ for all $\alpha \in \Phi$. Since $\Delta$ is a base of $\Phi$, then $\alpha = \sum_{i=1}^{l} k_{\alpha,i} \alpha_i$ where all $k_{\alpha,i}$ are integers. Therefore $\lambda = \sum_{i=1}^{l} k_i \alpha$, where $k_i = \sum_{\alpha \in \Phi} k_{\alpha,i} k_\alpha$. Since $\Phi$ is finite and all $k_{\alpha,i}$ are integers, then $k_i \in \mathbb{Z}$ for all $1 \leq i \leq l$.
Therefore $\lambda$ is in the $\mathbb{Z}$-span of $\Delta$. □

For a fixed base $\Delta$ of the root system $\Phi$, a weight $\lambda$ is called **dominant** if $<\lambda, \alpha> \geq 0$ for all $\alpha \in \Delta$. A weight $\lambda$ is called **strongly dominant** if $<\lambda, \alpha>> 0$ for all $\alpha \in \Delta$. Let $\Lambda^+$ denote the set of dominant weights. By definition, the fundamental Weyl chamber relative to $\Delta$, $\mathfrak{C}(\Delta)$, is the connected component of $E \setminus \cup_{\alpha\in\Phi} P_\alpha$ containing a regular $\gamma$ such that $\Delta = \Delta(\gamma)$ is the set of indecomposable elements of $\Phi^+ = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}$. Therefore $\Lambda^+$ is the set of weights lying in the closure of the fundamental Weyl chamber, and the set of strongly dominant weights is the intersection of the fundamental Weyl chamber with $\Lambda$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Let the basis dual (relative to the inner product) to $\Delta^\vee$ be denoted by $\{\lambda_1, \ldots, \lambda_l\}$. Then $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. An element of $\{\lambda_1, \ldots, \lambda_l\}$ is called a **fundamental dominant weight**. Indeed $\lambda_i$ is a weight, since $<\lambda_i, \alpha_j>= (\lambda_i, \alpha_j^\vee) = \delta_{ij} \in \mathbb{Z}_{\geq 0}$.

Note that, $\sigma_{\alpha_i}(\lambda_j) = \lambda_j - <\lambda_j, \alpha_i> \alpha_i = \lambda_j - \delta_{ij}\alpha_i$.

**Lemma 5:** $\Lambda$ is a lattice with basis $\Delta' = \{\lambda_1, \ldots, \lambda_l\}$. Furthermore, $\lambda \in \Lambda^+$ iff $m_i =<\lambda, \alpha_i> \geq 0$.

*Proof.* I will show that $\Lambda$ is the $\mathbb{Z}$-span of $\Delta'$.

Let $\sum_{i=1}^{l} k_i\lambda_i$ be in the $\mathbb{Z}$-span of $\Delta'$. By lemma 3, it is enough to show that $<\sum_{i=1}^{l} k_i\lambda_i, \alpha_j> \in \mathbb{Z}$. We have $<\sum_{i=1}^{l} k_i\lambda_i, \alpha_j>= \sum_{i=1}^{l} k_i <\lambda_i, \alpha_j>= \sum_{i=1}^{l} k_i\delta_{ij} = k_j \in \mathbb{Z}$.

Let $\lambda \in \Lambda$. Then $m_i =<\lambda, \alpha_i> \in \mathbb{Z}$ for all $1 \leq i \leq l$. Then $<\lambda - \sum_{i=1}^{l} m_i\lambda_i, \alpha_j>=<\lambda, \alpha_j> - \sum_{i=1}^{l} m_i <\lambda_i, \alpha_j>= m_j - \sum_{i=1}^{l} m_i\delta_{ij} = 0$ for all $1 \leq i \leq l$. Then $(\lambda - \sum_{i=1}^{l} m_i\lambda_i, \alpha_j) = 0$ for all $1 \leq j \leq l$. Thus $\lambda = \sum_{i=1}^{l} m_i\lambda_i$.
$\lambda \in \Lambda^+$ iff $<\lambda, \alpha_i> \geq 0$ for all $1 \leq i \leq l$ iff $m_i \geq 0$ for all $1 \leq i \leq l$. □

Examples:

(1) Calculate the fundamental dominant weights of $A_1$.
Let $\Phi = \{\pm\alpha_1\}$ be a root system pf $A_1$ with base $\{\alpha_1\}$. Let $\lambda_1 = k_1\alpha_1$. Since $<\lambda_1, \alpha_1>= \delta_{11} = 1$, then $2k_1 = 1$. Therefore $\alpha_1 = 2\lambda_1$.

(2) Calculate the fundamental dominant weights of $A_2$.
We know from previous chapters that the Cartan matrix of $A_2$ is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Say $\lambda_1 = k_1\alpha_1 + k_2\alpha_2$. Then

$$\begin{cases} k_1 < \alpha_1, \alpha_1 > +k_2 < \alpha_2, \alpha_1 >= 1 \\ k_1 < \alpha_1, \alpha_2 > +k_2 < \alpha_2, \alpha_2 >= 0 \end{cases} \quad \begin{cases} 2k_1 - k_2 = 1 \\ -k_1 + 2k_2 = 0 \end{cases} \quad \begin{cases} k_1 = \frac{2}{3} \\ k_2 = \frac{1}{3} \end{cases}$$

Thus $3\lambda_1 = 2\alpha_1 + \alpha_2$. Similarly, $3\lambda_2 = \alpha_1 + 2\alpha_2$.
The group $\Lambda/\Lambda_r$ is called **the fundamental group** of $\Phi$.

**Lemma 6:** The fundamental group of $\Phi$ has finite order equal to the determinant of the Cartan matrix of $\Phi$.

*Proof.* First I will prove the following: Let $M$ be a free $\mathbb{Z}$-module of rank $l$ with basis $\{y_1, \ldots, y_l\}$ and $L$ be a $\mathbb{Z}$-submodule of $M$ of rank $l$ with basis $\{x_1, \ldots, x_l\}$. Let $T$ denote the change of basis matrix from $\{y_1, \ldots, y_l\}$ to $\{x_1, \ldots, x_l\}$. Then the order of the group $M/L$ is $|detT|$.
We know from Algebra II, theorem 7.1, that there exists a basis $\{y'_1, \ldots, y'_l\}$ of $M$ and integers $m_1, \ldots, m_l$ such that $\{m_1y'_1, \ldots, m_ly'_l\}$ is a basis of $L$. Furthermore, the integers $m_i$ are unique up to multiplication by units and $m_1|m_2|\ldots|m_l$. Thus, w.l.o.g we can assume $m_i > 0$.
I will show next, that $\sum_{i=1}^{l} c_iy'_i$ where $0 \leq c_i \leq m_i - 1$ is a system of cosets representatives of $M/L$. By using the division algorithm, one can see that any element of $M$ is in one of these cosets, thus these cosets cover $M$. Since $0 \leq c_i \leq m_i - 1$, then $\sum_{i=1}^{l} c_iy'_i - \sum_{i=1}^{l} d_iy'_i \in L$ iff $(c_i - d_i)|m_i$ iff $c_i = d_i$. Therefore, all the cosets above are distinct. Then $|M/L| = [M : L] = m_1 \ldots m_l$.
Let $A$ be the change of basis matrix from $\{y_1, \ldots, y_l\}$ to $\{y'_1, \ldots, y'_l\}$, $B = diag(m_1, \ldots, m_l)$ the change of basis matrix from $\{y'_1, \ldots, y'_l\}$ to $\{m_1y'_1, \ldots, m_ly'_l\}$, and $C$ be the change of basis matrix from $\{m_1y'_1, \ldots, m_ly'_l\}$ to $\{x_1, \ldots, x_l\}$. Since $A$ and $C$ are invertible matrices with integer elements, then $detA, detC \in \{\pm 1\}$. Since $T = ABC$, then $|detT| = detB = m_1 \ldots m_n$. Therefore $|M/L| = |detT|$.

Say $\alpha_i = \sum_{i=1}^{l} m_{ij}\lambda_j$, where $m_{ij} \in \mathbb{Z}$. Then $< \alpha_i, \alpha_k >= \sum_{j=1}^{l} m_{ij} < \lambda_j, \alpha_k >= \sum_{j=1}^{l} m_{ij}\delta_{jk} = m_{ik}$. Thus the change of basis matrix from $\{\alpha_1, \ldots, \alpha_l\}$ to $\{\lambda_1, \ldots, \lambda_l\}$ is the Cartan matrix of $\Phi$. Then $|\Lambda/\Lambda_r|$ is the determinant of the Cartan matrix. □

Examples: (1) Calculate the determinant of the Cartan matrix of $A_l$.

We know from chapter 11, that the Cartan matrix of $A_l$ is

$$A_l = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

To calculate its determinant, we add all the columns to the first one and we expand the minors along the first column:

$$\begin{aligned} \begin{vmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{vmatrix} &= \begin{vmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 1 & 0 & 0 & 0 & \dots & -1 & 2 \end{vmatrix} \\ &= det(A_{l-1}) + (-1)^{l+1} \begin{vmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 2 & -1 \end{vmatrix} \\ &= det(A_{l-1}) + (-1)^{l+1}(-1)^{l-1} = det(A_{l-1}) + 1 \end{aligned}$$

We know that $A_1 = (2)$, thus $det(A_1) = 2$. By induction, it follows that $det(A_l) = l + 1$.

(2) Calculate the determinant of the Cartan matrix of $B_l$.
We know from chapter 11, that the Cartan matrix of $B_l$ is

$$B_l = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

Then $B_1 = (2)$ and thus $det(B_1) = 2$. We have $B_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ and, thus, $det(B_2) = 2$. To calculate the determinant we expand the minors along the first column.

$$
\begin{aligned}
B_l &= \begin{vmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} \\
&= 2\begin{vmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} \\
&= 2det(B_{l-1}) + (-1)\begin{vmatrix} 2 & -1 & \dots & 0 & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} \\
&= 2det(B_{l-1}) - det(B_{l-2})
\end{aligned}
$$

By induction, one can check that $det(B_l) = 2$.

(3) Calculate the determinant of the Cartan matrix of $C_l$.
We know from chapter 11, that the Cartan matrix of $C_l$ is

$$
C_l = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}
$$

Then $C_1 = (2)$ and thus $det(C_1) = 2$. We have $C_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ and, thus, $det(C_2) = 2$. A calculation similar to the one for $B_l$ gives the same recurrence relation, $det(C_l) = 2det(C_{l-1}) - det(C_{l-2})$. Similarly, it follows that $det(C_l) = 2$.

(4) Calculate the determinant of the Cartan matrix of $D_l$.
We know from chapter 11, that the Cartan matrix of $D_l$ is

$$
D_l = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 & 0 \\ & & & \dots & & & & & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & 2 \end{pmatrix}
$$

We have $D_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and, thus, $det(D_2) = 2$.

$$det(D_3) = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & -1 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = -2 + 6 = 4$$

A calculation similar to the one for $B_l$ gives the same recurrence relation, $det(D_l) = 2det(D_{l-1}) - det(D_{l-2})$. Similarly, it follows that $det(D_l) = 4$.

Note that the Weyl group $\mathcal{W}$ leaves $\Lambda$ invariant. Indeed if $\lambda$ is a weight then $< \lambda, \alpha > \in \mathbb{Z}$ for all $\alpha \in \Phi$. We know that an element of the Weyl group preserves the inner product and it permutes the elements of $\Phi$. Thus $\sigma\lambda$ has the desired property.

**Lemma 7:** Each weight is conjugate under $\mathcal{W}$ to one and only one dominant weight. If $\lambda$ is dominant, then $\sigma\lambda \prec \lambda$ for all $\sigma \in \mathcal{W}$, and if $\lambda$ is strongly dominant, then $\sigma\lambda = \lambda$ only when $\sigma = 1$.

*Proof.* To prove this lemma we will use exercise 10.14 and lemma 10.3B.
Recall lemma 10.3B: Let $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$. If $\sigma\lambda = \mu$ for some $\sigma \in \mathcal{W}$, then $\sigma$ is a product of simple reflections which fix $\lambda$; in particular $\lambda = \mu$.
I will prove next exercise 10.14: prove that each point of $E$ is $\mathcal{W}$-conjugate to a point in the closure of the fundamental base chamber relative to a base $\Delta$.
Recall $\beta \prec 0$ if $\beta = \sum_{i=1}^{l} k_i\alpha_i$ where all $k_i \leq 0$ and $\mu \prec \lambda$ iff $\mu - \lambda \prec 0$. Let $\mu \in E$. Let $\sigma \in \mathcal{W}$ such that $\lambda = \sigma\mu$ is maximal with the property that $\mu \prec \lambda$. Assume $\lambda \notin \overline{\mathfrak{C}(\Delta)}$. Then there exists $\alpha_i \in \Delta$ such that $< \lambda, \alpha_i >< 0$. Then $\sigma_{\alpha_i}\lambda - \lambda = - < \lambda, \alpha_i > \alpha_i$. Thus $\lambda \prec \sigma_{\alpha_i}\lambda$. This contradicts the choice of $\lambda$. Therefore $\lambda \in \overline{\mathfrak{C}(\Delta)}$.
Let $\lambda$ be a weight. By the exercise 10.14, there exists $\lambda' \in \overline{\mathfrak{C}(\Delta)}$ $\mathcal{W}$-conjugate to $\lambda$. Since $\Lambda$ is closed under the action of the Weyl group, then $\lambda'$ is also a weight, thus a dominant weight.
Assume $\lambda$ is conjugate to two dominant weights, $\lambda'$ and $\lambda''$. Then $\lambda' \in \overline{\mathfrak{C}(\Delta)}$ and $\lambda'' \in \overline{\mathfrak{C}(\Delta)}$. By theorem 10.3B, $\lambda' = \lambda''$.
If $\lambda$ is dominant, then $\lambda \in \overline{\mathfrak{C}(\Delta)}$. Let $\mu \in \overline{\mathfrak{C}(\Delta)}$ be the $\mathcal{W}$-conjugate of $\sigma\lambda$ by the reflection $\tau$. Then $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$ with $\tau\sigma\lambda = \mu$. By the above result $\sigma\lambda \prec \mu$ and by lemma 10.3B $\mu = \lambda$. Therefore $\sigma\lambda \prec \lambda$.
Assume $\lambda$ is strongly dominant and $\sigma\lambda = \lambda$. Then $\lambda \in \mathfrak{C}(\Delta)$. From 10.1, we know that $\Delta(\sigma\lambda) = \sigma\Delta(\lambda)$, and thus $\Delta = \sigma(\Delta)$. By theorem 10.3e, $\sigma = 1$. □

Note that the following is possible: $\mu$ is dominant, $\lambda$ is not dominant and $\mu \prec \lambda$.

**Lemma 8:** Let $\lambda \in \Lambda^+$. Then the number of dominant weights $\mu \prec \lambda$ is finite.

*Proof.* Let $\mu$ be a dominant weight such that $\mu \prec \lambda$. Since $\mu$ is a dominant weight, then $< mu, \alpha > \geq 0$ for all $\alpha \in \Delta$. Since $\lambda \in \Lambda^+$, then $< \lambda, \alpha > \geq 0$ for all $\alpha \in \Delta$.

Then $< \lambda + \mu, \alpha >\geq 0$ for all $\alpha \in \Delta$. Since $\mu \prec \lambda$, then $\lambda - \mu = \sum_{i=1}^{l} k_i \alpha_i$ with $k_i \geq 0$ for all $1 \leq i \leq l$. Thus $(\lambda + \mu, \lambda - \mu) = \sum_{i=1}^{l} k_i(\lambda + \mu, \alpha_i) \geq 0$. Therefore $(\lambda, \lambda) - (\mu, \mu) \geq 0$. Thus $\mu \in \{(x, x) \in E : (x, x) \leq (\lambda, \lambda) \cap \Lambda^+$. Since then set $\{(x, x) \in E : (x, x) \leq (\lambda, \lambda)$ is compact and $\Lambda^+$ is discrete, then their intersection is finite. □

**Lemma 9:** Let $\delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha$. Then $\delta = \sum_{j=1}^{l} \lambda_j$, so $\delta$ is a strongly dominant weight.

*Proof.* Recall corollary to lemma 10.2B: $\sigma_\alpha \delta = \delta - \alpha$ for all $\alpha \in \Delta$.
Since $\sigma_\alpha \delta = \delta - < \delta, \alpha > \alpha$, then $< \delta, \alpha >= 1$ for all $\alpha \in \Delta$. We showed already that $\delta = \sum_{i=1}^{l} < \delta, \lambda_i > \lambda_i$. Thus $\delta = \sum_{j=1}^{l} \lambda_j$.
By lemma 5, $\delta \in \Lambda^+$. □