Main Theorem

Let $L$ and $L'$ be two topologically trivial Legendrian knots in a tight contact 3-manifold. If $tb(L) = tb(L')$ and $r(L) = r(L')$ then $L$ and $L'$ are Legendrian isotopic.
Proof Strategy

Let \( L \) be a Legendrian knot bounding an embedded disk \( D \).
1. Perturb the foliation
2. Build a tree
3. Define a front projection and a foliation
4. Modify the tree

Catalog of Wavefronts

- \( r=-s < 0, \ tb = -(2t+1+s) \)
Catalog of Wavefronts

• \( r = s > 0, \ tb = -(2t+1+s) \)
  – Reverse orientations in the previous slide
• \( r = 0, \ tb = -(2t+1) \)

Step1: Perturb the foliation

Goal: Given a spanning disk \( D \) of \( L \), perform a \( C^0 \)-small perturbation of \( D \) to obtain a spanning disk \( D' \) of \( L \) with foliation in elliptic form.
1. Just \( h^+ \) and \( e^- \) on boundary
2. Just \( h^+ \) and \( e^- \) on boundary and just \( e^+ \) and \( h^- \) on interior
3. Mostly \( h^+ \) and \( h^- \) on boundary, just \( e^+ \) and \( e^- \) on interior
Elliptic Foliation

- Signs of boundary singularities alternate
- Boundary singularities connect only with their direct neighbors on the boundary and interior singularities
- All interior singularities are elliptic
- Interior singularities connect to at least two boundary hyperbolic singularities
Just h+ and e- on boundary

• If $tb(L) = t$ then there is a $C^0$-small perturbation of $D$ such that there are exactly $2t$ singularities on the boundary and they have alternating signs.
• Elliptic-hyperbolic conversion

Just h- and e+ on interior

• Destroy hyperbolic-hyperbolic connections
• Eliminate negative elliptic singularities
• Eliminate positive hyperbolic singularities
Just e- and e+ on interior (1)

Just e- and e+ on interior (2)
Just e- and e+ on interior (3)

Just e- and e+ on interior (4)
Step 2: Build a Tree

- Skeleton of the foliation
  - Vertices - interior elliptic points
  - Edges – representative arcs
- Extended skeleton of the foliation
  - New vertices – elliptic boundary points
  - New edges – representative arcs
- Signed trees
- Have an acceptable planar embedding
Build an wavefront

- Choose disjoint neighborhoods of vertices
- Leftmost vertex

- End vertex

- Otherwise – replace the subtree to the right by a reflection of it in the horizontal axis

Recap

- Start with Legendrian knot \( L \) spanned by the embedded disk \( D \)
- Perturb \( D \) to have an elliptic foliation
- Get an embedded Legendrian tree \( T \) (extended skeleton)
- Given a planar embedding of \( T \) build a front projection \( W_T \)

Claim: The lift of \( W_T \) bounds an embedded disk whose foliation is elliptic and diffeomorphic to the elliptic foliation of \( D \).
Forget about L (1)

Suppose Legendrian knots $L$ and $L'$ bound $D$ and $D'$ with diffeomorphic characteristic foliations in elliptic form. Then $L$ and $L'$ are Legendrian isotopic.

Convert the elliptic form spanning disk to exceptional form spanning disk

Forget about L (2)

Isotopy supported in the complement of small neighborhood of end vertices.
Forget about L (3)

Use Elliptic Pivot Lemma to extend the isotopy to the entire disk.

We can assume that in a neighborhood of the elliptic point we can choose cylindrical coordinates $(\rho, \phi, z)$ and the contact form is $dz + \rho^2 d\phi$. Let $L_c$ be a piecewise-smooth Legendrian curve in the horizontal plane consisting of two rays $\phi = 0$ and $\phi = c$.

For any $\epsilon > 0$ there exists a Legendrian isotopy $\hat{L}_c$, $c \in (0, \pi]$ such that $\hat{L}_\pi = L_\pi$ and for all $c \in (0, \pi]$ the curve $\hat{L}_c$ coincides with $L_c$ outside of the $\epsilon$-neighborhood of the origin.

Step 4: Modify the Tree
Step 4: Modify the Tree